# Stochastic Processes, Nonlocal Operators, and Partial Differential Equations
in MIMIC

Woojin Ko

Korea Advanced Institute of Science and Technology

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# Overview

# Definition of Brownian Motions

**Definition.**

A collection $\{X_t : t \geqslant 0\}$ of random vectors $X_t : \Omega \to \mathbb{R}^n$ is called a ($n$-dimensional) stochastic process.

**Definition.**

Let $B = \{B_t : t \geqslant 0\}$ be a $n$-dimensional stochastic process. We say $B$ is a Wiener process if

(a) $B_0 = 0$;

(b) (independence of increments) for any $t_1 \leqslant t_2 \leqslant t_3$, $B_{t_2} - B_{t_1}$ and $B_{t_3} - B_{t_2}$ are independent;

(c) (Gaussian increments) $B_{t_2} - B_{t_1} \sim N(0, (t_2 - t_1)\Sigma)$;

(d) the map $t \mapsto B_t$ is continuous;

We also call $B$ the Brownian motion.

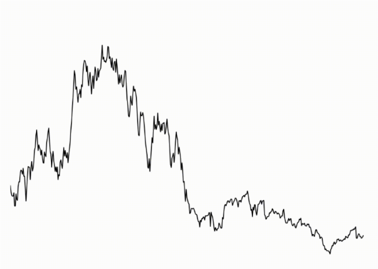
# Properties of Brownian Motions

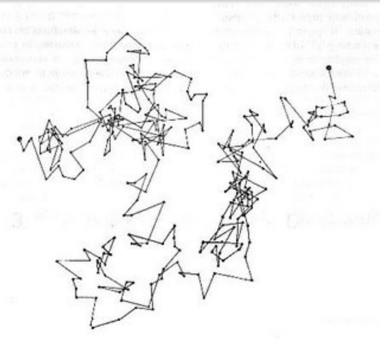
## Properties

Let $B = \{B_t : t \geqslant 0\}$ be a Brownian motion.

(i) (Scailing Relation) $B_{st} \overset{\mathrm{d}}{=} \sqrt{t} B_s$.

(ii) (Hölder Continuouity) $t \mapsto B_t \in C^{0,\gamma}$ for all $\gamma < 1/2$.





# Definition of Lévy Processes

**Definition.**

Let $X = \{X_t : t \geqslant 0\}$ be a ($n$-dimensional) stochastic process. We say $X$ is a Lévy process if

(a) $X_0 = 0$;

(b) (independence of increments) for any $t_1 \leqslant t_2 \leqslant t_3$, $X_{t_2} - X_{t_1}$ and $X_{t_3} - X_{t_2}$ are independent;

(c) (stationarity of increments) $X_{t_2} - X_{t_1} \overset{\mathrm{d}}{=} X_{t_2 - t_1}$;

(d) (continuity in probability) for any $\epsilon > 0$ and $t \geqslant 0$, we have

$$\lim_{h \to 0+} P(|X_{t+h} - X_t| > \epsilon) = 0.$$

# Characterization of Lévy Processes

## Properties

Let $X = \{X_t : t \geqslant 0\}$ be a Lévy process.

(i) (Lévy-Khintchine) There exist a vector $b \in \mathbb{R}^n$, a positive semidefinite real matrix $\Sigma$, and a finite meausre $\nu$ satisfying $E[e^{i\xi \cdot X_t}] = e^{-t\psi(\xi)}$ where

$$\psi(\xi) = -i\xi\cdot b+\xi^t\Sigma\xi+\int_{\mathbb{R}^n\setminus\{0\}}\left(1 - e^{i\xi\cdot y} + i\xi \cdot y1_{\{|y|\leqslant 1\}}\right)\nu(dy).$$

We say that $\psi$ is the characteristic exponent of $X_t$. Also, the triplet $(b, \Sigma, \nu)$ is uniquely determined.

# Characterization of Lévy Processes

**Properties (continued)**

(ii) (Decompostion) $X_t$ can be uniquely decomposed as

$$X_t = bt + B_t + J_t,$$

where $bt$ represent a drift, $B_t$ is the Brownian motion satisfying $B_t \sim N(0, 2t\Sigma)$, and $J_t$ is a right-continuous pure jump process satisfying $E[\#\{J_t - J_{t-} \in A : t \leqslant 1\}] = \nu(A)$ for all measurable $A \subset \mathbb{R}^n$. Also, these 3 processes are mutually independent.

# Examples

(1) **The Standard Brownian motion** $B_t$ is a Lévy process with the triplet $(0, I, 0)$.

(2) **The Poisson process** $N_t$ with rate $\lambda$ which describes, for example, the number of phone call between time 0 and $t$ is a Lévy process with $b = 0$ and $\Sigma = 0$.

(3) **Subordinate Brownian Motion**:
Let $S_t$ be a 1-dimensional non-decreasing Lévy process, that is, it corresponds to a triplet $(b, 0, \nu)$ where $b \geqslant 0$ and supp$\nu \subset \mathbb{R}_{\geqslant 0}$. For example, the Poisson process $N_t$ in (2) may be $S_t$. Then, $B_{S_t}$ is a Lévy process where $B$ is a Brownian motion.

# Motivation

Now, we want to decribe behaviors of particles that follow a stochastic process, in particular a Lévy process.
Assume that particles are spread out following the distribution density $f : \mathbb{R}^n \to \mathbb{R}$ at $t = 0$. Consider each particle moves (also may jump) as a Lévy process $X$.

- $Z$: a random vector independent to $X_t$ with $Z \sim f$ (after normalizing);
- $X_t \sim \mu_t$.

Then, $Z + X_t \sim f * \mu_t = \int f(\cdot - y)\mu_t(dy)$.
If $\mu_t$ is symmetric, the distribution of particles at time $t$ has the density

$$x \mapsto \int f(x+y)\mu_t(dy) = E[f(X_t + x)] =: T_t f(x).$$

# Motivation

A natural proceeding is to differentiate $T_t f$. Roughly, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} T_t f(x) &= \lim_{h \to 0+} \frac{E[f(X_{t+h} + x)] - E[f(X_t + x)]}{h} \\
&= \lim_{h \to 0+} \frac{E[E[f(X_t + X_h + x)]] - E[f(X_t + x)]}{h} \\
&= \lim_{h \to 0+} \frac{E[T_t f(X_h + x)] - T_t f(x)}{h}.
\end{aligned}
$$

Note that $X_{t+h} \overset{\mathrm{d}}{=} X_t + X_h$ by the stationarity of increments of Lévy process.

# Definition of Infinitesimal Generators

**Definition**

Let $X_t$ be a $n$-dimensional Lévy process. For a suitable function $u : \mathbb{R}^n \to \mathbb{R}$, define $Au : \mathbb{R}^n \to \mathbb{R}$ by

$$Au(x) := \lim_{h \to 0+} \frac{E[u(X_h + x)] - u(x)}{h}, \quad x \in \mathbb{R}^n.$$

We say $A$ is the (infinitesimal) generator of $X_t$.

Then, $u(x,t) := T_t u_0(x) = E[u_0(X_t + x)]$ is a solution, actually a unique one, of the following equation:

$$\begin{cases} u_t = Au & \text{in } \mathbb{R}^n \times [0,\infty); \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^n \end{cases}$$

where $A$ is the generator of $X_t$. Therefore, we note that each particle follows $A$ infinitesimally.

# Computation of $A$

## Definition

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be the characteristic exponent of a Lévy process of the form

$$\psi(\xi) = -i\xi \cdot b + \xi^t \Sigma \xi + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{i\xi \cdot y} + i\xi \cdot y 1_{\{|y| \leqslant 1\}}\right) \nu(dy).$$

Define an operator $\psi(D)$ by

$$\begin{aligned}\psi(D)u(x) := &-b \cdot \nabla u(x) - (\nabla^t \Sigma \nabla) u(x) \\ &+ \int_{\mathbb{R}^n \setminus \{0\}} \left(u(x) - u(x+y) + y \cdot \nabla u(x) 1_{|y| \leqslant 1}\right) \nu(dy).\end{aligned}$$

# Computation of $A$

**Theorem**

Let $X_t$ be a Lévy process, and $\psi$, $A$ be its characteristic exponent and generator, repectively. Then $A = -\psi(D)$.

**Idea of proof.** Use the Fourier transform on $T_t f$.

# Brownian Motion and Heat Equation

Let $B_t$ be a $n$-dimensional Brownian motion with $B_t \sim N(0, 2tI)$. Then the generator of $B_t$ is $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, the Laplacian operator in $\mathbb{R}^n$. And the solution of the equaion

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times [0, \infty); \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n \end{cases}$$

is given by

$$u(x,t) = E[u_0(B_t + x)] = u_0 * K_t(x) = \int_{\mathbb{R}^n} u_0(y) \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} dy$$

where $K_t(x) = e^{-|x|^2/4t}/(4\pi t)^{n/2}$ is a Gaussian function which is the density of $\sim N(0, 2tI)$.
The above's are well-knowned as a study of "Heat Equation."

# Local and Nonlocal Operators

## Definition

Let $L : F \to G$ be an operator for some function spaces $F = F(X)$ and $G = G(X)$ with domain $X$.
We say that $L$ is local if for any $x \in X$ and any neighborhood $N$ of $x$, we have $Au(x) = A(u1_N)(x)$.
We say $L$ is nonlocal if it is not local.

## Examples

- Every differential operator $\partial_{x_j}$ on $\mathbb{R}^n$ is a local operator.
- The Fourier transform $\mathcal{F} : u \mapsto \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u(x) dx$ and its inverse $\mathcal{F}^{-1}$ are nonlocal operators.

# Nonlocality of Generators

Let $X_t$ be a Lévy process with the characteric exponent $\psi$.
For convinience,

$$\psi(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{i\xi \cdot y}\right) \nu(dy).$$

Then,

$$Au(x) = -\psi(D)u(x) = \int_{\mathbb{R}^n \setminus \{0\}} \left(u(x+y) - u(x)\right) \nu(dy).$$

Since $\nu(dy) = E[\#\{X_t - X_{t-} \in dy : t \leqslant 1\}]$, we can also guess that $Au$ is the infinitesimal change of amount of particles distributed as $u$ which behave like $X_t$.
Note that $A$ is a nonlocal operator since it needs all of informations over $\mathbb{R}^n$.

One of the most important example of nonlocal operator is the fractional Laplacian.

## Definition

Let $s \in (0,1)$. For a suitable function $u : \mathbb{R}^n \to \mathbb{R}$, define

$$
\begin{aligned}
(-\Delta)^s u(x) &:= C(n,s) P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \\
&= C(n,s) \lim_{\epsilon \to 0+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy
\end{aligned}
$$

where $C(n,s) = \left( \int_{\mathbb{R}^n} \frac{1-\cos \zeta_1}{|\zeta|^{n+2s}} d\zeta \right)^{-1}$ is a constant depending only on $n$ and $s$. We say $(-\Delta)^s$ is the fractional Laplacian operator of order $s$.

## Proposition

For suitable $u$, we have

$$(-\Delta)^s u = \mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}u].$$

Since $-\Delta u = \mathcal{F}^{-1}[|\cdot|^2\mathcal{F}u]$, it is the reason of that the operator is called as "the fractional Laplacian".
Also, the fractional Laplacian can be computed as

$$-(-\Delta)^s u(x) = C(n,s)\int_{\mathbb{R}^n\setminus\{0\}} \frac{u(x+y)-u(x)}{|y|^{n+2s}}dy.$$

Therefore, $-(-\Delta)^s$ is the infinitesimal generator of a Lévy process $X_t$ with triplet $(0,0,\nu)$ where $\nu(dy) = C(n,s)|y|^{-n-2s}dy$ if such a process exists.

## Existence

Let $\alpha = 2s \in (0,2)$. There exists a Lévy process $X_t$ such that its characteristic exponent $\psi$ is given by

$$\psi(\xi) = |\xi|^\alpha = \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{i\xi \cdot y}\right) \frac{C(n,s)}{|y|^{n+\alpha}} dy.$$

The process $X_t$ is often called the $\alpha$-stable process. The $\alpha$-stable process $X_t$ has the characteristic function

$$E[e^{i\xi \cdot X_t}] = e^{-t\psi(\xi)} = e^{-t|\xi|^\alpha}$$

and it is a unique process which satisfies

$$X_t^{(1)} + \cdot + X_t^{(m)} \overset{\mathrm{d}}{=} m^{1/\alpha} X_t$$

where $X_t^{(j)}$'s denote the independent copies of $X_t$. Also, $X_t$ is a subordinate Brownian motion, i.e. there is $S_t$ such that $X_t = B_{S_t}$.

# Parabolic and Elliptic PDEs

In the previous section, we considered the equation

$$u_t = Lu$$

for some linear operator $L$. When $L = \Delta$, we get the heat equation. We can regard the heat equation as finding points in some space satisfying

$$\partial_t = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2,$$

while paraboloid in $\mathbb{R}^{n+1}$ is of the form

$$t = c_1 x_1^2 + \cdots + c_n x_n^2, \quad (x_1, \cdots, x_n, t) \in \mathbb{R}^{n+1}.$$

$$u_t = Lu + \text{(additional term)}: \text{ “Parabolic Type”}$$

# Parabolic and Elliptic PDEs

## Examples

- $u_t - \Delta u = 0$ (Heat Equation)
- $u_t - \Delta u = f(u)$ (Scalar Reaction Diffusion Equation)
- $iu_t + Lu = 0$ (Schrödinger Equation)
- $iu_t + Lu = f(|u|^2)u$ (Nonlinear Schrödinger Equation)

# Parabolic and Elliptic PDEs

Consider that it's been a long enogh time and the system is stable with almost equilibrium state. Then,

$$Lu = u_t = 0.$$

When $L = \Delta$, we get the harmonic equation. We can regard the equation as finding points in some space satisfying

$$\partial_{x_1}^2 + \cdots + \partial_{x_n}^2 = 0,$$

while ellipsoid in $\mathbb{R}^n$ is of the form

$$c_1 x_1^2 + \cdots + c_n x_n^2 = c, \quad (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

$$Lu + (\text{additional term}) = 0\text{: “Elliptic Type”}$$

# Parabolic and Elliptic PDEs

## Examples

- $-\Delta u = 0$ (Harmonic Equation)
- $-\Delta u = f(u)$ (Nonlinear Poisson Equation)
- $Lu = \lambda u$ (Eigenvalue problem of Elliptic Equation)
- $\nabla \cdot (d\nabla u) + u(m-u) = 0$ (Logistic Equation)

# Elliptic PDE with Nonlocal Operator

- $X_t$: Lévy Process;
- $\psi$: Characteristic Exponent of $X_t$;
- $A = -\psi(D)$: Generator of $X_t$.

We may consider the following elliptic equation with nonlocal operator:

$$Au + (\text{additional term}) = 0.$$

For example,

$$(-\Delta)^{\alpha/2} u = f(u)$$

for some $f$ which may be nonlinear. (e.g. $f(u) = |u|^{p-1}u$)
This equation describes the equilibrium state of particles that follows the process with external force.

# My Research

**Question**

- $X_t$: Subordinate Brownian Motion;
- $\psi$: Characteristic Exponent of $X_t$;
- $f : \mathbb{R} \to \mathbb{R}$ : continuous on $(0, \infty)$, $f(t) = 0$ for $t \leqslant 0$, $\lim_{t \to 0+} f(t) = -\infty$.

For example, $f(t) = \log t$ for $t > 0$.
Does a (radially symmetric nonnegative) solution of following equation exist?

$$\psi(D)u = f(u) \quad \text{in } \mathbb{R}^n$$

# Interesting Results (Liouville Theorem)

**Theorem**

Let $s \in (0,1)$ and consider

$$(-\Delta)^s u = 0 \quad \text{in } \mathbb{R}^n.$$

Then, $u(x) = a \cdot x + b$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
Furthermore, $a = 0$ if $s \leqslant 1/2$.

**Theorem**

Let $\psi$ be the characteristic exponent of a Lévy process with a finite moment and $\{\xi : \psi(\xi) = 0\} = \{0\}$, and consider

$$-\psi(D)u = 0 \quad \text{in } \mathbb{R}^n.$$

Then, $u$ must be a polynomial.

# Thank you!